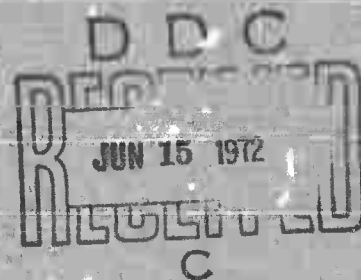Division of Engineering
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AN INFINITE SLAB OF ISOTROPIC, TRANSVERSELY INHOMOGENOUS ELASTIC MATERIAL SUBJECTED TO TENSION

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An Infinite Slab of Isotropic, Transversely Inhomogenous Elastic Material Subjected to Tension*

by

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Abstract

The dependence of the stress field on a continuous periodically varying stiffness which approximates a layered material is studied. This dependence is compared to the solution for the usual approximation of "converting" this class of problems to the superposition of an anisotropic one and a residual boundary problem. The region of applicability of the usual approximation is found to be as expected geometrically but surprisingly large as far as variation of stiffness is concerned.

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Introduction

Inhomogeneities in stiffness of isotropic elastic materials occur in several areas of technology. In particular various applications in rock mechanics, layered materials and composite materials are related directly to such inhomogeneities [1,2,3]*. When an elastic modulus varies periodically with one Cartesian coordinate the usual situations occuring in rock mechanics and layered materials can be represented. This work is concerned with such a case.

Let the wavelength of elastic modulus variation be L and some characteristic dimension for a particular boundary value problem be c. When $L \ll c$ an analytical procedure has been described [2] for "converting" the inhomogenous problem into the superposition of two problems which are 1) an anisotropic material problem with the original configuration and boundary conditions and 2) a residual boundary problem whose effect on the stress distribution should be important only a few wavelengths into the body from the surface.

This superposition technique is currently the standard procedure for dealing with this class of problems. The reason is that more solutions and techniques [4] are available for solving anisotropic problems than for inhomogenous problems. Also, the residual boundary problem can be approximated as a local one independent of the configuration of the particular problem under consideration. Presumably this inhomogenous residual boundary problem can be solved "once and for all" for a particular type of stiffness variation and inclination of the boundary to the layers.

The residual boundary problem has received some attention in the literature. When the variation of the stiffness is discontinuous (e.g., bonded, layered materials) the residual boundary problem has been considered by

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*Numbers in brackets refer to the References at the end of the paper.

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Bogy [5]. When the stiffness variation is sinusoidal with small amplitude compared to the mean and the boundary is perpendicular to the layers the solution to the residual boundary problem may be found in [6]. In [5,6] a strong dependence of the stresses on Poisson's ratio is found.

The object of this work was to solve an elementary problem with continuously varying, large, stiffness changes without resorting to the superposition technique described above. By solving this inhomogenous elastic problem of an infinite slab under tension perpendicular to the layers it was hoped the region of applicability of the superposition technique could be more firmly established. It is best to have an exact solution to an appropriate inhomogenous problem for this purpose but none are currently available.

Figure 1 shows the problem considered here. The infinite slab has a thickness $2c$ and the wavelength for the shear modulus variation is L. The problem is two dimensional so that a very elementary numerical procedure may be used to approximate values for the stresses. The solution for the dimensionless stresses σ_{xx}/σ_{AVG}, σ_{yy}/σ_{AVG} and σ_{xy}/σ_{AVG} depends on the following constant parameters:

ν = Poisson's ratio

L/c = Material property variation wavelength/characteristic dimension of the body

ϵ = A measure of the stiffness variation

The practical ranges of these parameters chosen here for study are

$0.1 \leqslant \nu \leqslant 0.5$

$0.1 \leqslant L/c \leqslant 3.0$

$0 \leqslant \epsilon \leqslant 0.7$

In the following two sections the problem is formulated and an approximate solution technique outlined. Then the solution presented in [6] which

employs the superposition of anisotropic and residual boundary solutions is given for the problem of Figure 1. Subsequently numerical results are presented comparing the approximate solution obtained here to the perturbation solution from [6] and conclusions are drawn.

Formulation of the Problem

Since the shear modulus $G(x)$ is periodic in x the stresses σ_{ij} are also periodic. With the notation

$$f(x) = \{1 - \epsilon\cos(2\pi x/L)\}^{-1} \tag{1}$$

the relevant plane strain stress-strain relations are

$$
\begin{bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \end{bmatrix}
=
\frac{f(x)}{2G_o}
\begin{bmatrix} (1-\nu) & -\nu & 0 \\ -\nu & (1-\nu) & 0 \\ 0 & 0 & 2 \end{bmatrix}
\cdot
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}
\tag{2}
$$

where u and v are the x and y displacements. The stresses σ_{xx}, σ_{yy} and σ_{xy} can be deduced from the Airy stress function $\psi(x,y)$ so that they satisfy the force equilibrium equations for this static plane strain problem as

$$\sigma_{xx} = \psi,_{yy} \quad , \quad \sigma_{yy} = \psi,_{xx} \quad \text{and} \quad \sigma_{xy} = \psi,_{xy} \tag{3}$$

while σ_{zz} is given by $\nu(\sigma_{xx} + \sigma_{yy})$ and σ_{xz} as well as σ_{yz} vanish. Combining equations (2) and (3) to eliminate u and v yields

$$g(\psi) \equiv \psi,_{xxxx} + 2\psi,_{xxyy} + \psi,_{yyyy} + 2\,\frac{f,_x}{f}\,(\psi,_{xx} + \psi,_{yy}),_x$$

$$+ \frac{f,_{xx}}{f}\,(\psi,_{xx} - \frac{\nu}{1-\nu}\,\psi,_{yy}) = 0 \tag{4}$$

Appropriate boundary conditions for this problem are deduced as follows:

σ_{xy}, σ_{yy} vanish on $y = \pm c \rightarrow \psi,_{xx}(x,\pm c) = 0$, $\psi,_{xy}(x,\pm c) = 0$

no net shear force on cross-section $\rightarrow \psi,_x(x,-c) = \psi,_x(x,+c)$

no net moment on cross-section $\rightarrow c\{\psi,_y(x,c) + \psi,_y(x,-c)\} = \psi(x,c)-\psi(x,-c)$

net axial force/depth is $2\sigma_{AVG}c \rightarrow \psi,_y(x,c) - \psi,y(x,-c) = 2\sigma_{AVG}c$

$$(5)$$

It is noted that the usual step of replacing ν in the plane strain formulation by $\frac{\nu}{1+\nu}$ will yield the formulation for plane stress.

Outline of Solution

The following series approximation for ψ in terms of the $(m-1)n$ arbitrary constants $H_{i,j}$ $(i = 2,3...m; j = 1,2...n)$ was chosen

$$\psi = \frac{1}{2} \sigma_{AVG} (y^2 - c^2) + \sum_{k=1}^{m/2} \sum_{q=1}^{n} H_{2k;q} \{\cos(\pi ky/c) - (-1)^k\} \cos(2\pi qx/L)$$

$$+ \sum_{k=1}^{m/2-1} \sum_{q=1}^{n} H_{2k+1;q} \{\cos[(2k+1)\pi y/2c] - (-1)^k(2k+1)\cos(\pi y/2c)\} \cos(2\pi qx/L)$$

$$\equiv \frac{1}{2} \sigma_{AVG} (y^2 - c^2) + \sum_{\ell=2}^{m} \sum_{q=1}^{n} H_{\ell;q} h_{\ell;q} (x,y) \qquad (6)$$

ψ from equation (6) satisfies all the boundary conditions given in equation (5). The $H_{\ell;q}$ are determined using the $h_{\ell;q}(x,y)$ as weighting functions so that

$$\int_{x=0}^{L} \int_{y=-c}^{+c} h_{\ell;q} g(\psi) \, dx \, dy = 0 \qquad \ell = 2,3...m; q = 1,2...n \qquad (7)$$

when ψ from equation (6) is substituted into (7). The resulting $(m-1)n$ simultaneous equations for the $H_{i;j}/L^2\sigma_{AVG}$ depend on L, ϵ and ν and, of

course, m and n . These equations were programmed and solved on a digital computer for about 150 specific choices of L, ε, ν, m and n . The $H_{i;j}/L^2\sigma_{AVG}$ were subsequently used to find σ_{ij}/σ_{AVG} by means of equations (3) and (6). In addition to the dimensionless stresses σ_{xx}/σ_{AVG}, σ_{yy}/σ_{AVG} and σ_{xy}/σ_{AVG} the dimensionless, plane strain, octahedral, shear stress τ/σ_{AVG} was evaluated from

$$\tau/\sigma_{AVG} = \{\tfrac{1}{3}(1-\nu+\nu^2)[(\sigma_{xx}/\sigma_{AVG})^2 + (\sigma_{yy}/\sigma_{AVG})^2] - \tfrac{1}{3}(1+2\nu-2\nu^2)(\sigma_{xx}/\sigma_{AVG})(\sigma_{yy}/\sigma_{AVG})$$

$$+ (\sigma_{xy}/\sigma_{AVG})^2\}^{1/2} \tag{8}$$

These dimensionless stresses were evaluated for combinations of x/L and y/c of interest. Values of m and n were increased in trial runs and corresponding variations in τ_{max}/σ_{AVG} for fixed L, ε and ν were observed. It appears that an accuracy of a few percent can be expected using m and n equal to 4 in the range of L, ε and ν considered. Due to the large number of cases computed it was necessary to use the minimum values of m and n which yield results of acceptable accuracy.

Perturbation Solution

Nusayr and Paslay [6] recently determined the stresses near the boundary for the material described in Figure 1 using a perturbation technique. The stresses near the boundary in the present problem, therefore, should match the solution in [6] when the restrictions of [6] apply. These restrictions are that L/c and ε be small compared to one. The solution is

$$\sigma_{xx}/\sigma_{AVG} = 1-\varepsilon\{\nu/(1-\nu)\}\{1-(2\pi c/L)[1-(y/c)]\}e^{-(2\pi c/L)[1-(y/c)]}\cos(2\pi x/L)$$

$$\sigma_{yy}/\sigma_{AVG} = -\varepsilon\{\nu/(1-\nu)\}\{1+(2\pi c/L)[1-(y/c)]\}e^{-(2\pi c/L)[1-(y/c)]}\cos(2\pi x/L) \qquad (9)$$

$$\sigma_{xy}/\sigma_{AVG} = +\varepsilon\{\nu/(1-\nu)\}(2\pi c/L)[1-(y/c)]e^{-(2\pi c/L)[1-(y/c)]}\sin(2\pi x/L)$$

where $\varepsilon \ll 1$, $L/c \ll 1$, $0 \leqslant x \leqslant L/2$ and $0 \leqslant y \leqslant c$. In this paper equations (9) are referred to as the perturbation solution.

Numerical Results

In the above it was pointed out that attention here was restricted to ranges of the parameters of $0.1 \leqslant \nu \leqslant 0.5$, $0 \leqslant \varepsilon \leqslant 0.7$ and $0.1 \leqslant L/c \leqslant 3.0$. These ranges cover the region of most practical interest. Computations were made for enough combinations of these parameters to indicate trends in the solutions and to provide curves suitable for applications.

Frequently the most important quantity for a designer to have is the maximum value of a yield criterion. In this case the plane strain octahedral shear stress criterion was evaluated and designated τ . For each computed solution the maximum value of τ/σ_{AVG} was chosen from a grid of computed values over the region $0 \leqslant x \leqslant L/2$ and $0 \leqslant y \leqslant c$. In every case the maximum value of τ/σ_{AVG} was on the $x = L/2$ boundary of this region. Figure 2 shows results obtained from the computations. The three plots for different values of ν show the strong dependence, mentioned above, of the solution on Poisson's ratio. When $\nu = 0.1$ the influence of the considered ε and L/c variations on τ_{max}/σ_{AVG} is negligible. When $\nu = 0.5$ the influence of increasing ε from 0 to 0.7 can almost double the value for the yield criterion τ_{max}/σ_{AVG}. The influence of increasing L/c is to decrease the severity of the change of

τ_{max}/σ_{AVG} with ϵ as one intuitively might expect. The curve for $\nu = 0.3$ shows behavior intermediate to those for 0.1 and 0.5. In Figure 2 the dashed curves show the predictions of the perturbation solution, equations (9). As long as $L/c \leqslant 1.0$ the perturbation solution gives an acceptably accurate prediction for τ_{max}/σ_{AVG}.

An effect in this solution that can be easily comprehended is the development of σ_{yy} due to differential lateral contraction at $x = 0$ and $x = L/2$. A tensile σ_{yy}/σ_{AVG} develops at $x = 0$, $y = 0$ while a compressive σ_{yy}/σ_{AVG} developes at $x = L/2$, $y = 0$. These are the minimum and maximum values of σ_{yy}/σ_{AVG} in the region $0 \leqslant x \leqslant L/2$ and $0 \leqslant y \leqslant c$. In each computed case the value of the maximum σ_{yy} was greated than the absolute value of the minimum σ_{yy}. The perturbation solution gives $\sigma_{yy\ max} = |\sigma_{yy\ min}|$. Figure 3 shows results for $\sigma_{yy\ max}$ in the same format as Figure 2. It is apparent from these curves that the Poisson's ratio influence is present, that $\sigma_{yy\ max}$ is the order of $2\epsilon\nu\sigma_{AVG}$ for $L/c < 1.0$ and that the perturbation solution gives reasonable estimates for $\sigma_{yy\ max}$ over the range of parameters considered.

Conclusions

The results of the analysis presented here help to make clear the influence of isotropic inhomogeneities in isotropic elastic materials. the numerical results in Figures 2 and 3 indicate the influence on τ_{max}/σ_{AVG} and $\sigma_{yy\ max}/\sigma_{AVG}$ of Poisson's ratio ν, the inhomogeneity wavelength-characteristic length ratio L/c and the inhomogeneity strength parameter ϵ. The applicability of a perturbation solution which approaches the solution to the problem as $L/c \to 0$ and $\epsilon \to 0$ is shown to have a wider applicability in the solution to the problem considered here than the author originally anticipated. It is not

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surprising that the perturbation solution fails to adequately predict τ_{max}/σ_{AVG}
for $L/c > 1$ however the range of ϵ values for which τ_{max}/σ_{AVG} and
$\sigma_{yy\ max}/\sigma_{AVG}$ are, for most purposes, adequately predicted is surprisingly
large.

Acknowledgment

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<u>Figure Captions</u>

Figure 1   Configuration and description of material properties for the
            problem under consideration.

Figure 2   Dependence of the maximum plane strain octahedral shear stress
            τ_{max}/σ_{AVG} on the material variation parameter ε, the material
            wavelength/characteristic length parameter L/c and Poisson's
            ratio ν. Solid lines give the solution presented here while
            dashed lines give the perturbation solution from [6] which is
            independent of L/c.

Figure 3   Dependence of the maximum $|\sigma_{yy}|$, $\sigma_{yy\ max}$, on the material
            variation parameter ε, the material wavelength/characteristic
            length parameter L/c and Poisson's ratio ν. Solid lines give
            the solution presented here while dashed lines give the perturba-
            tion solution from [6].

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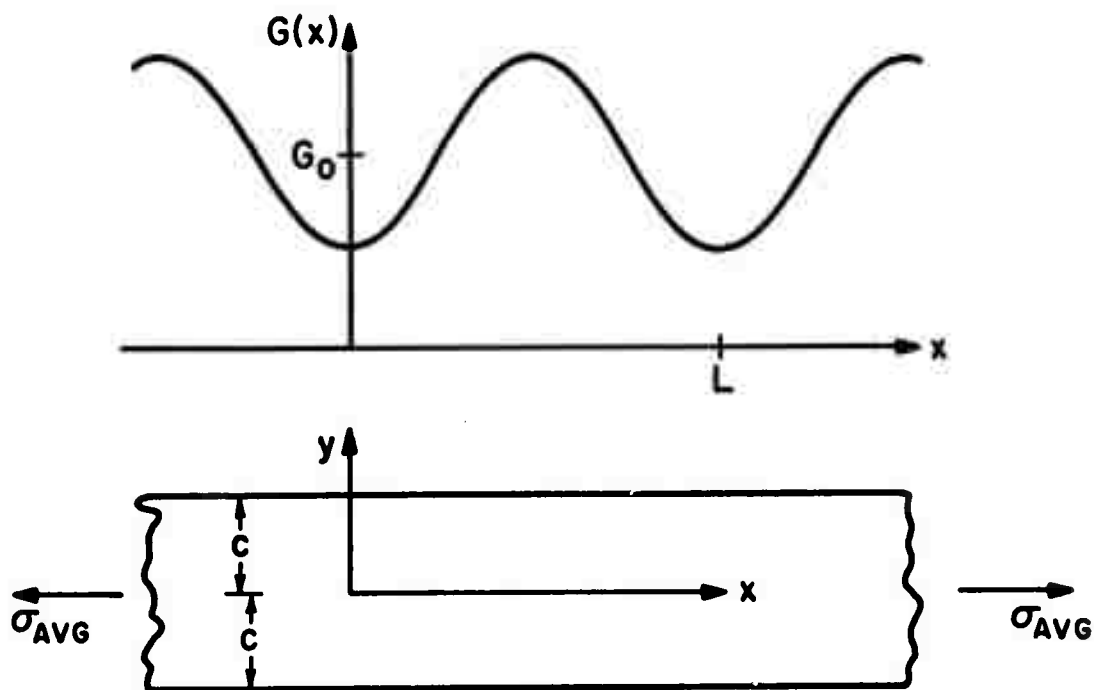
$G(x) = $ SHEAR MODULUS $= G_0 \left(1 - \epsilon \cos 2\pi \dfrac{x}{L}\right), \ 0 < \epsilon < 1$

$\nu = $ CONSTANT POISSON'S RATIO

$\sigma_{AVG} = $ AVERAGE σ_{xx} STRESS ON CROSS-SECTION

PLANE STRAIN

FOR $\nu = 0.5$

$\sigma_{yy\,max}/\sigma_{Avg}$

$\frac{L}{C} = 0.1$

0.3

1.0

0.1–1.0

3.0

FOR $\nu = 0.3$

$\sigma_{yy\,max}/\sigma_{Avg}$

$\frac{L}{C} = 0.1$

0.3

1.0

0.1–1.0

3.0

FOR $\nu = 0.1$

$\sigma_{yy\,max}/\sigma_{Avg}$

$\frac{L}{C} = 0.1$–0.3

1.0

0.1–1.0

3.0

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